Assembly of Finite Element Methods on Graphics Processors

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Institute for Mathematics and its Applications

1. **Motivation**
2. **FEM Assembly**
3. **GPU Assembly**
   - LocalElem
   - GlobalNZ
   - SharedNZ
   - Scatter and Reduction Arrays
4. **Results**
5. **Application**
6. **Conclusion**
7. **FMM GPU**

Domain Specific Languages

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  - Mesh-based PDEs on Heterogeneous Platforms
  - Analyzable DSL: Language with domain-specific features and restrictions that provide context for domain specific transformations

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- Stencils statically analyzed from access patterns

Why FEM Assembly on the GPU?

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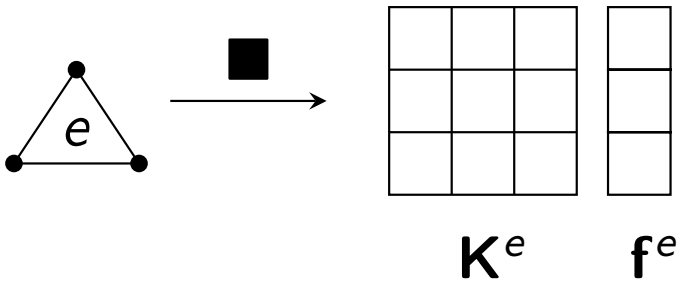
- Can assemble, solve, update, and visualize on the GPU
  - Completely avoid transfers with CPU.
  - Fast (real-time) simulations with visualization.

FEM Direct Assembly

Most common FEM assembly procedure:

- Compute element data.
  - One by one.



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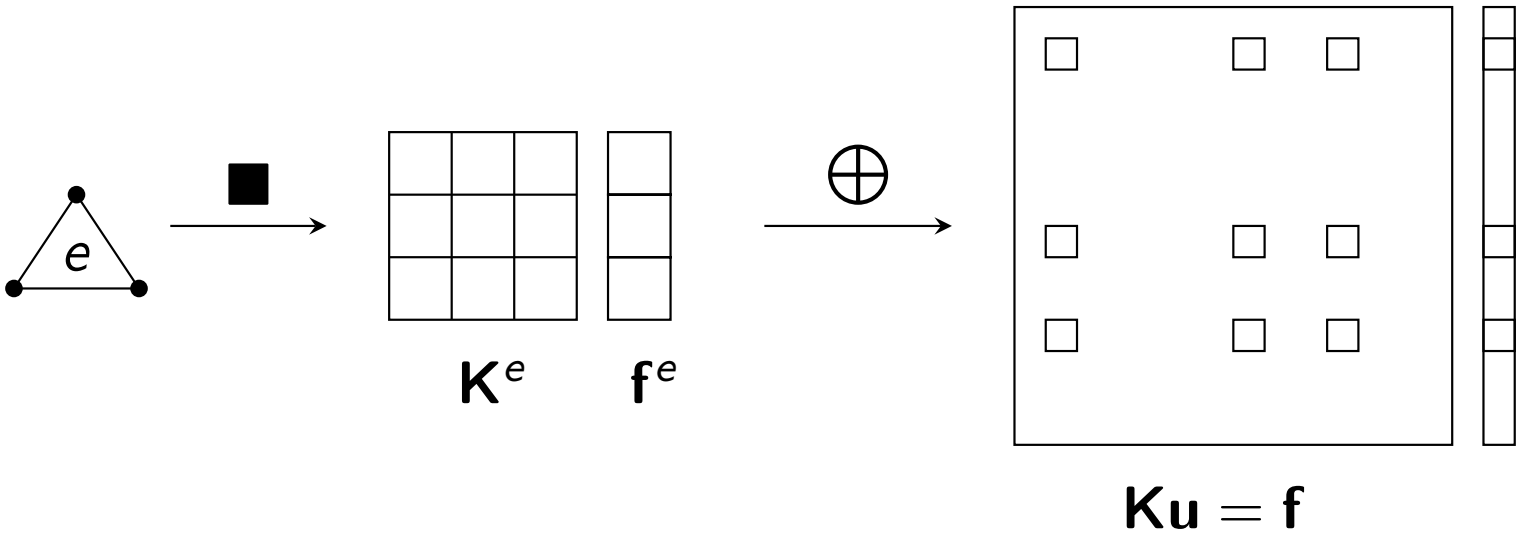
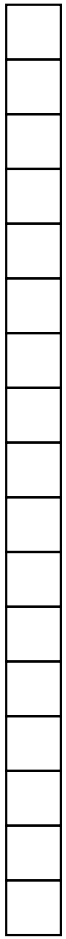
- Accumulate into global system.
  - Using a local index to global index mapping.



Data Flow

Nodal Data



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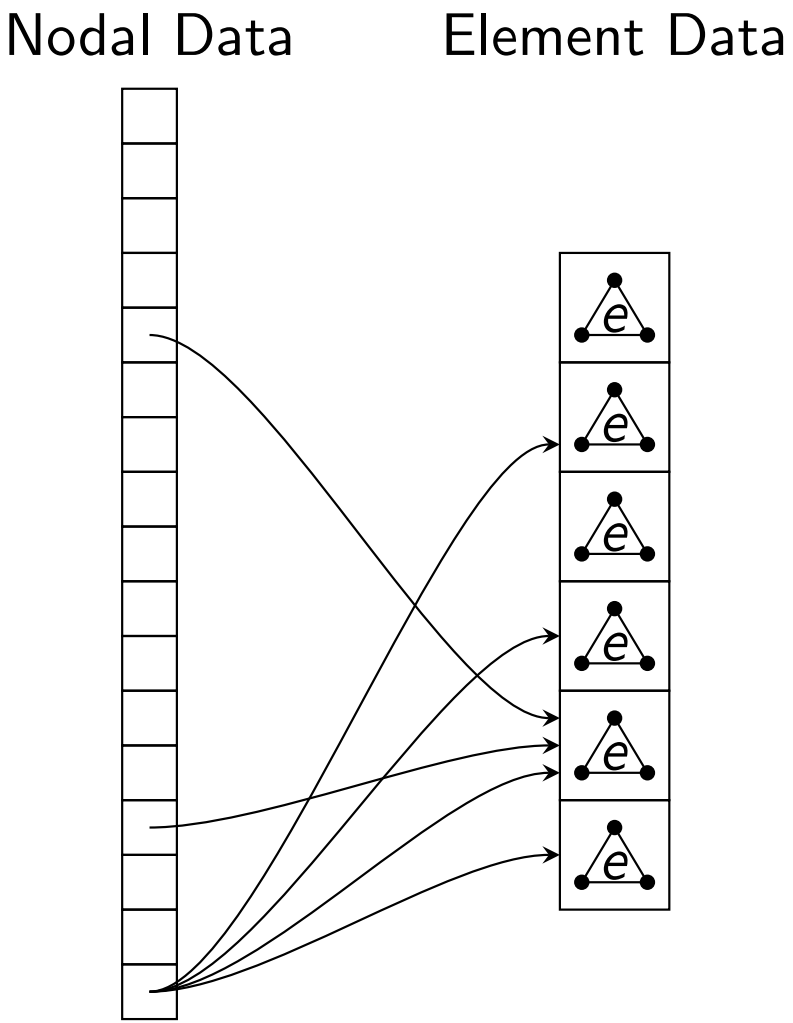

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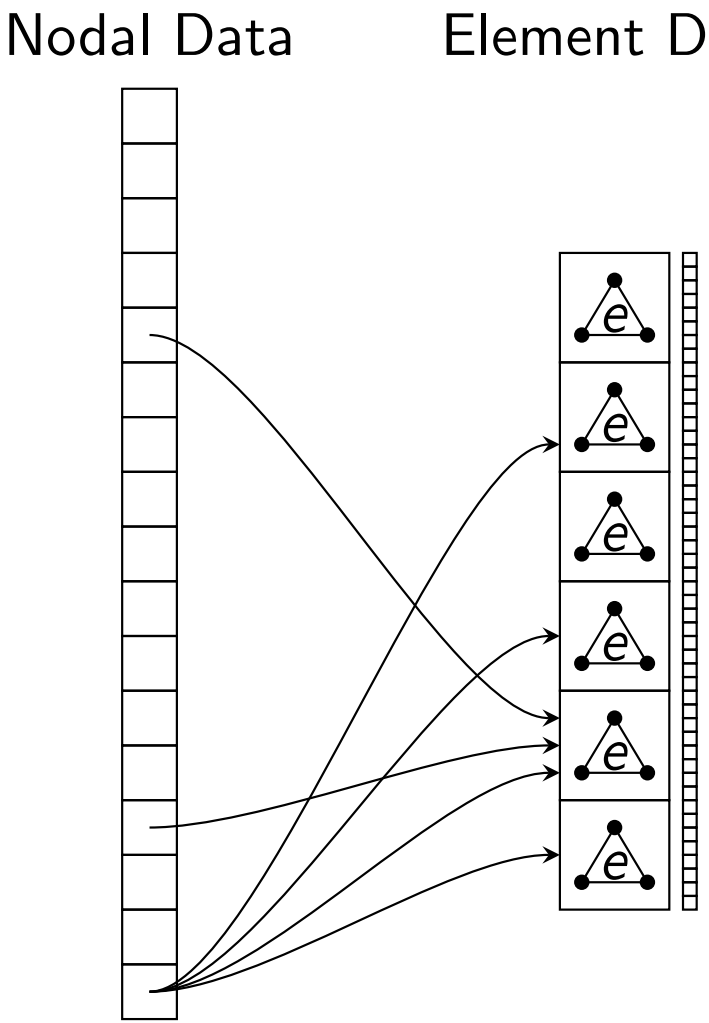

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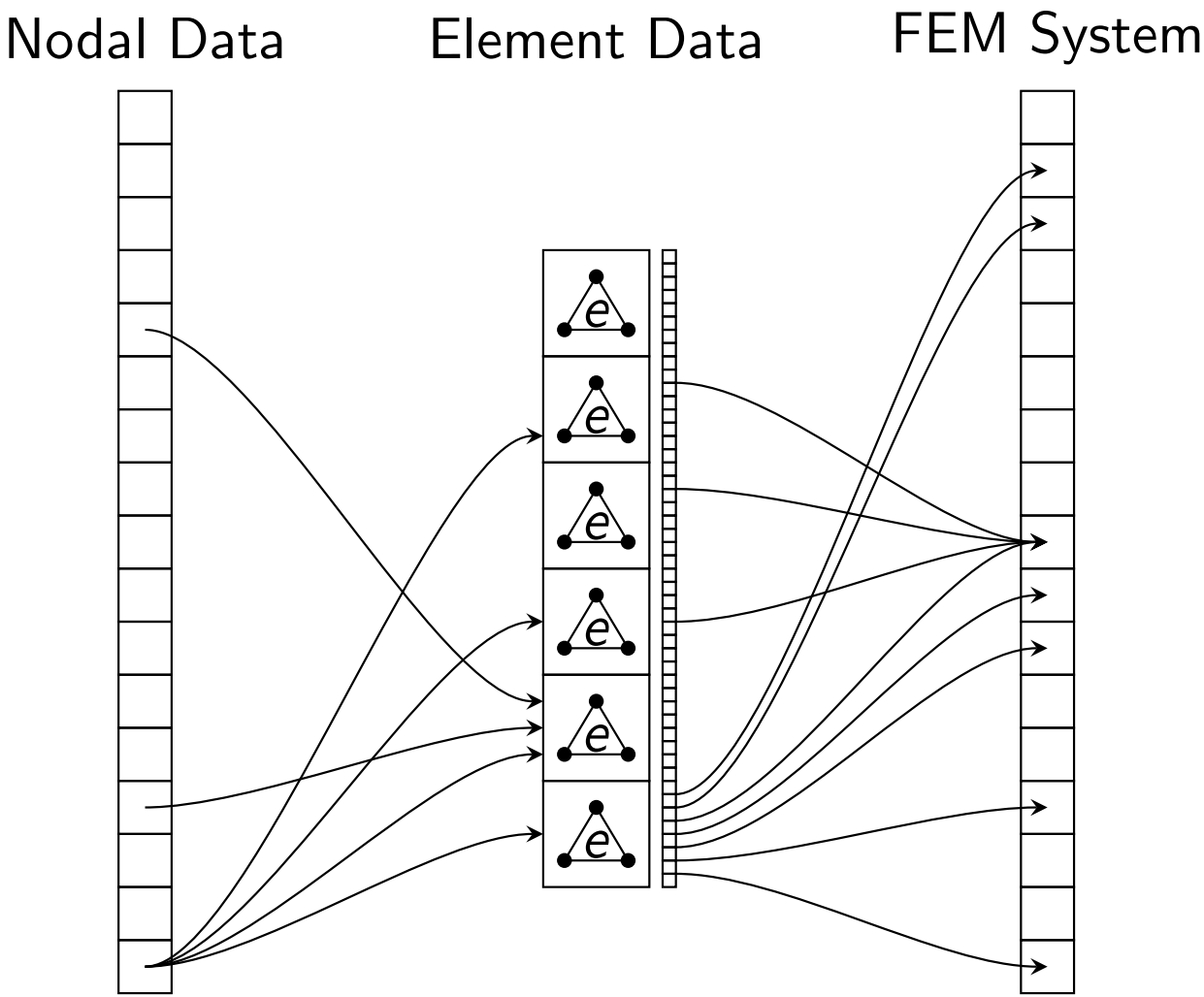

GPU FEM Assembly Strategies

Two Key Choices:

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Threads Assemble By

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  - Race conditions.

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Race conditions still possible!

Local-Element - Coloring the Mesh

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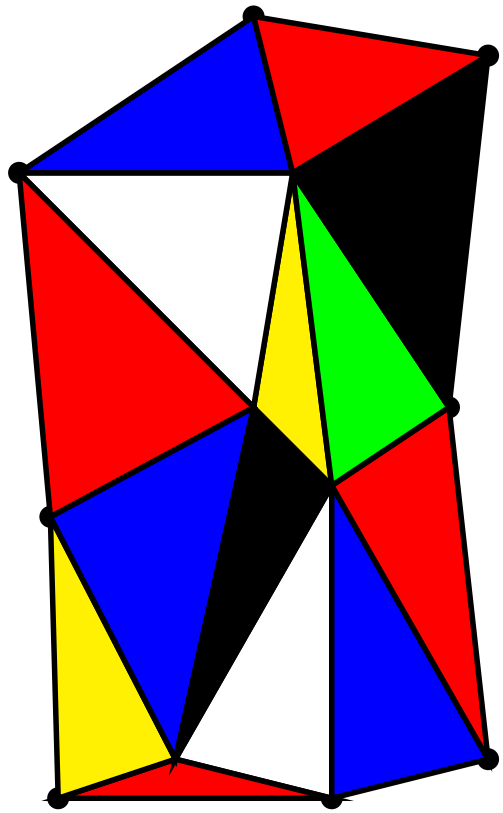
Partition elements to resolve race conditions.

- Transform into a coloring problem.



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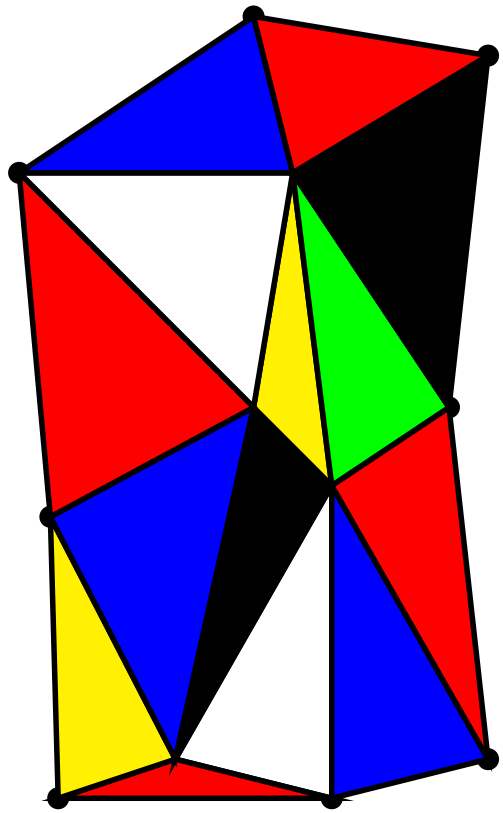
Problems.

- No sharing of nodal or element data.
- Little utilization of GPU resources.



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- Cluster the elements so they share nodes.
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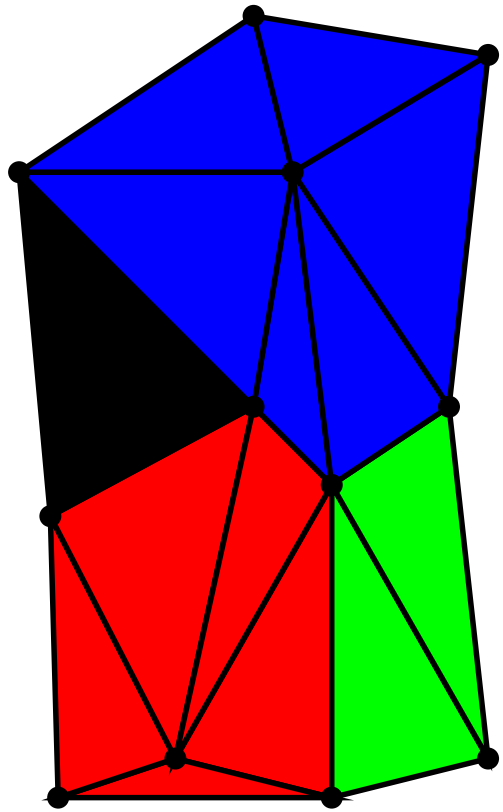

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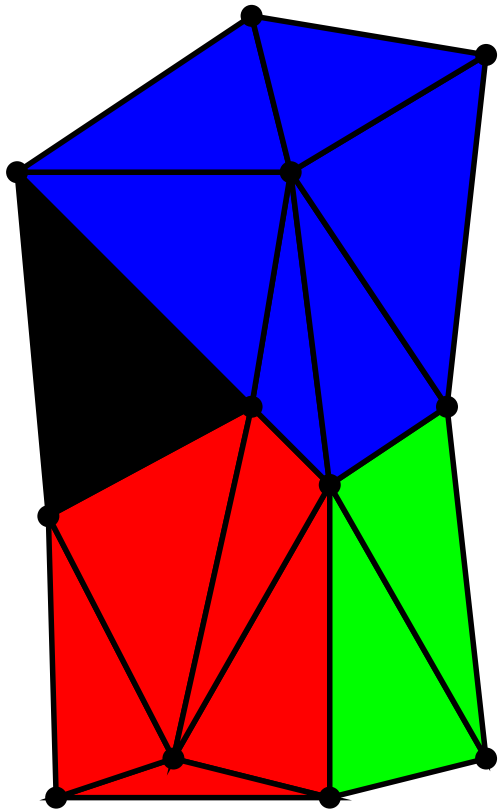
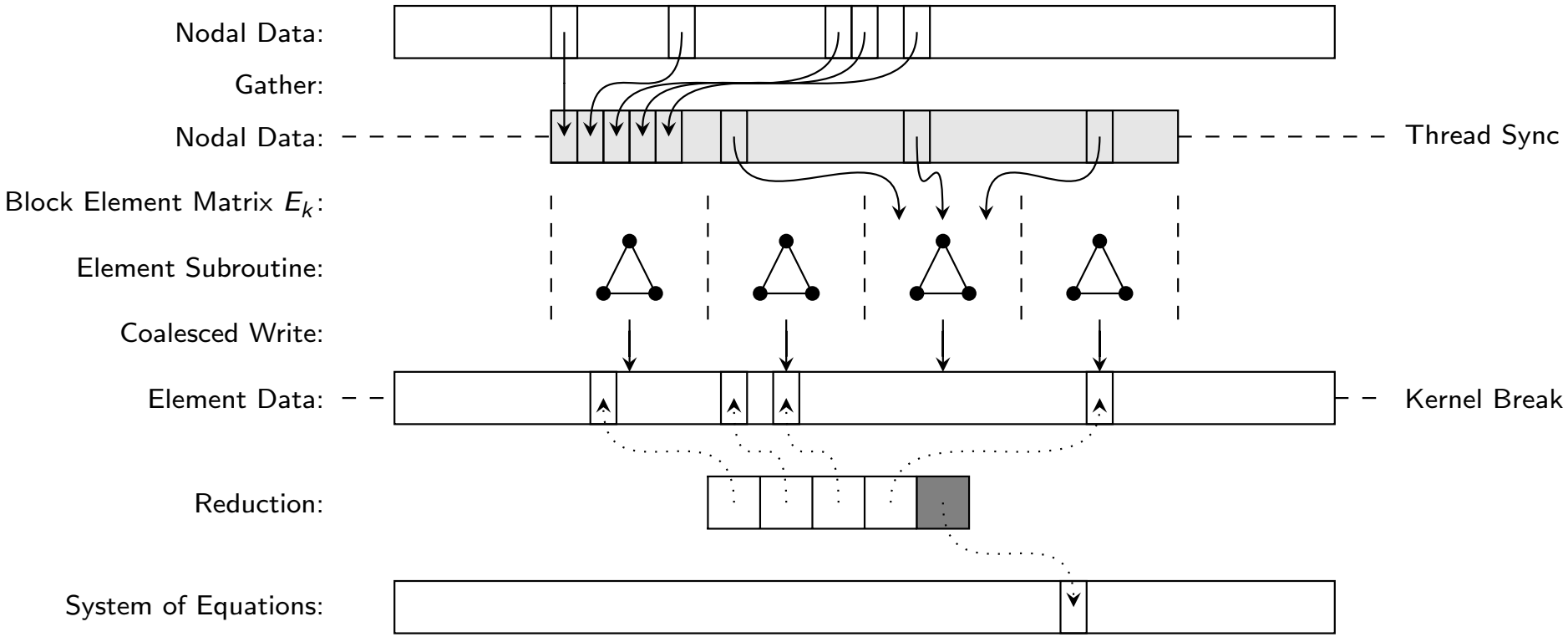


Global-NZ Data Flow

The optimized algorithm looks like:



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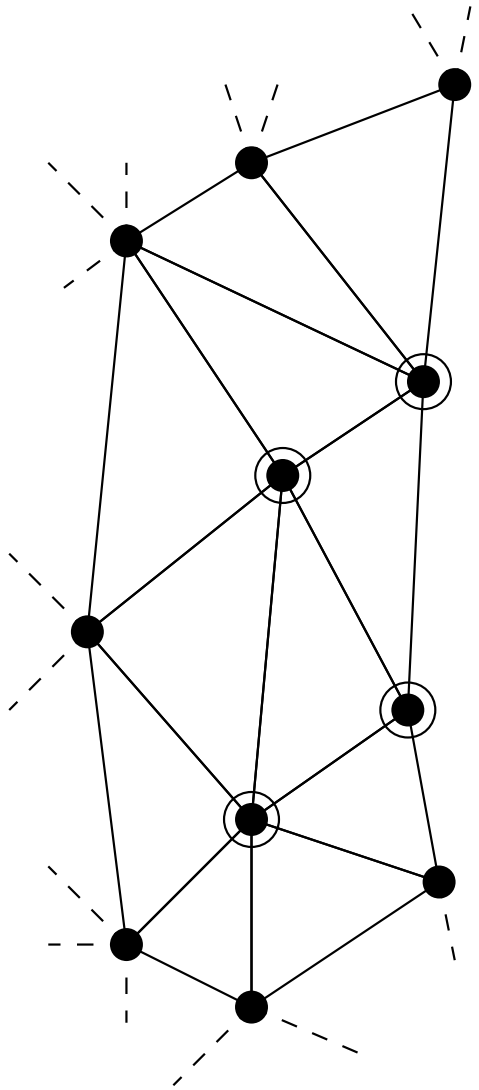

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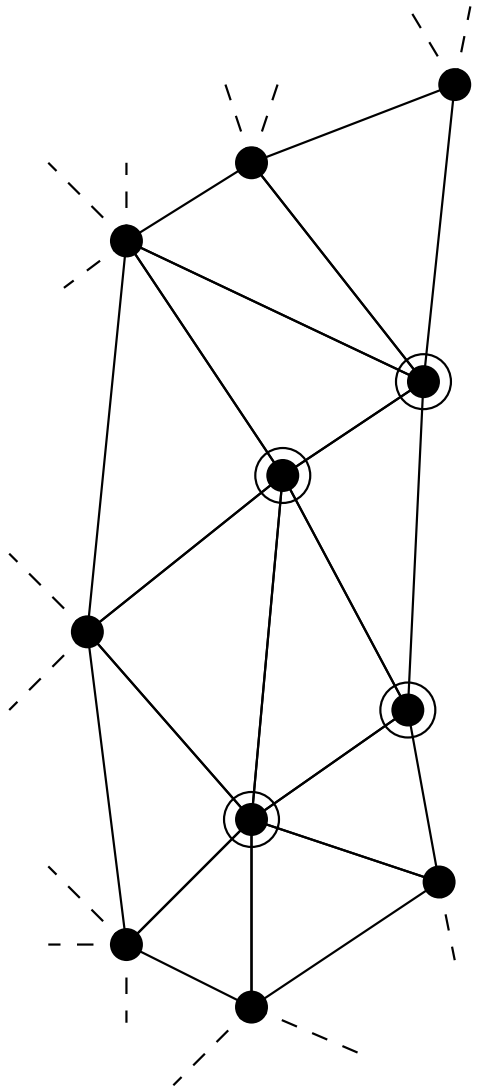
Problems.

- Shared memory size is very limiting.



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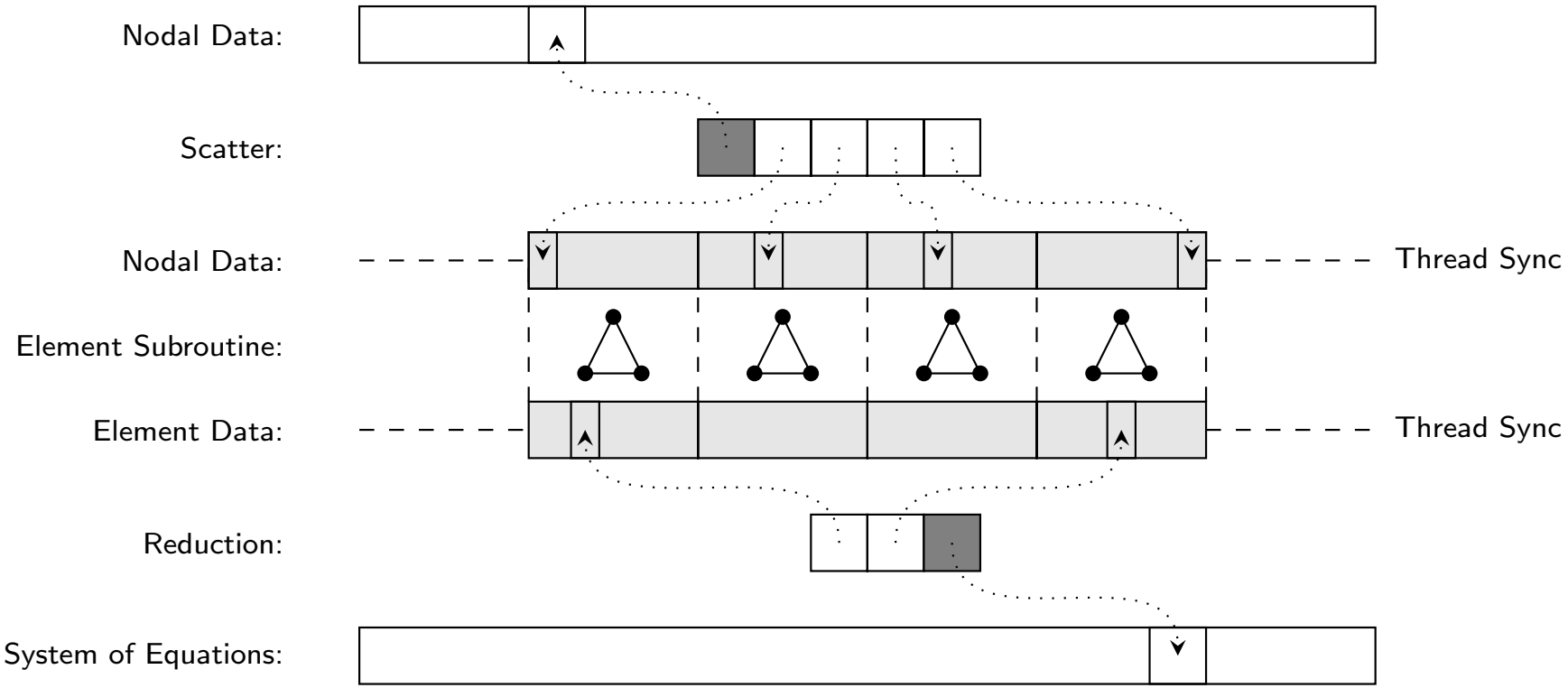

Scatter and Reduction Arrays

General procedure:

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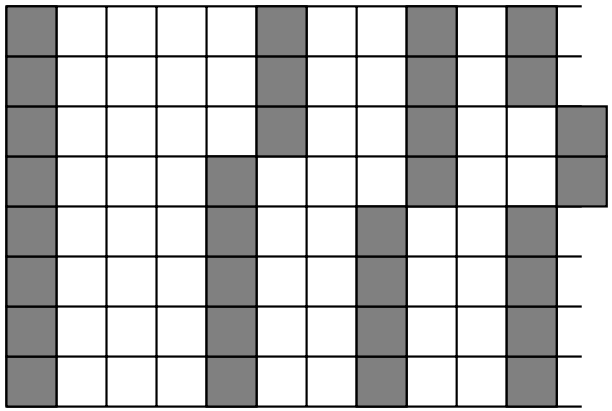

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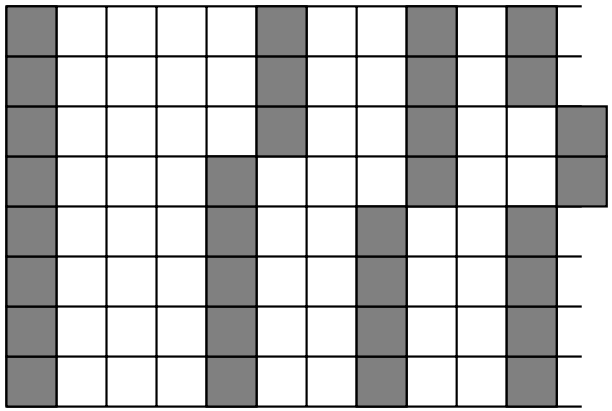

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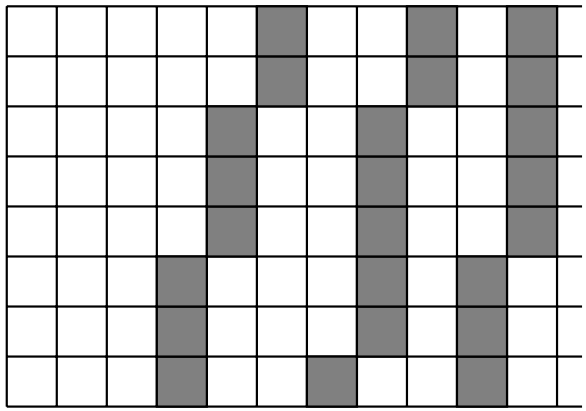

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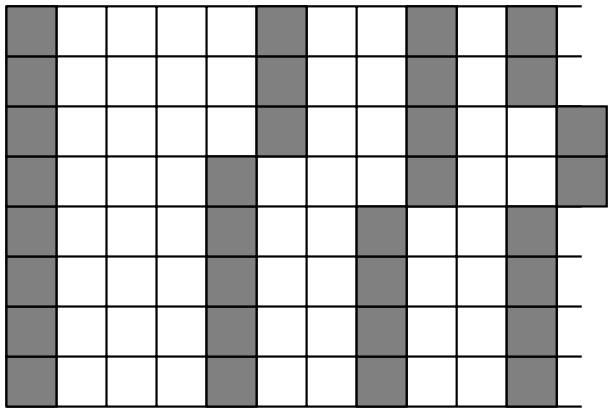

- Very fast.
- Highly adaptable.

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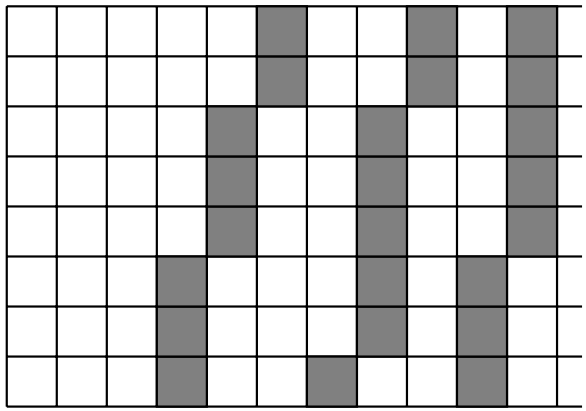
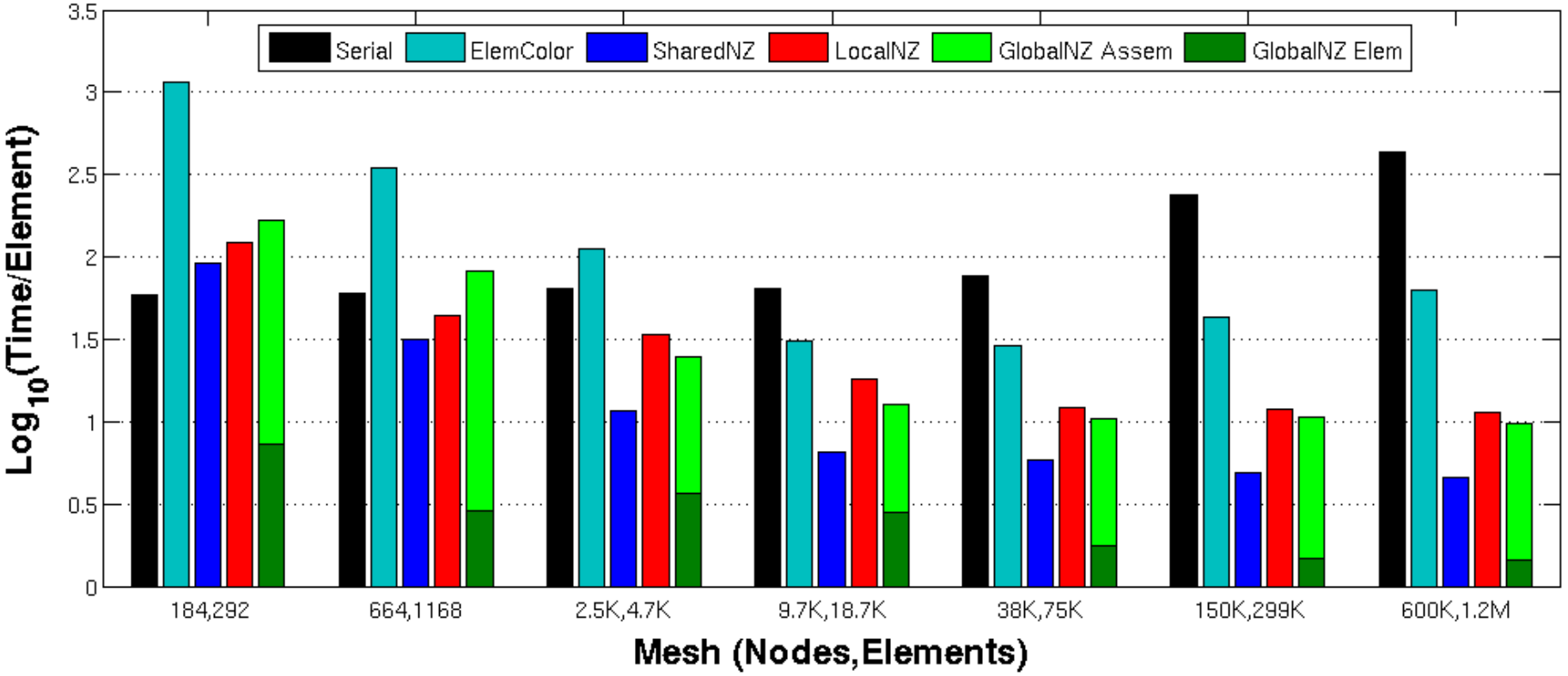
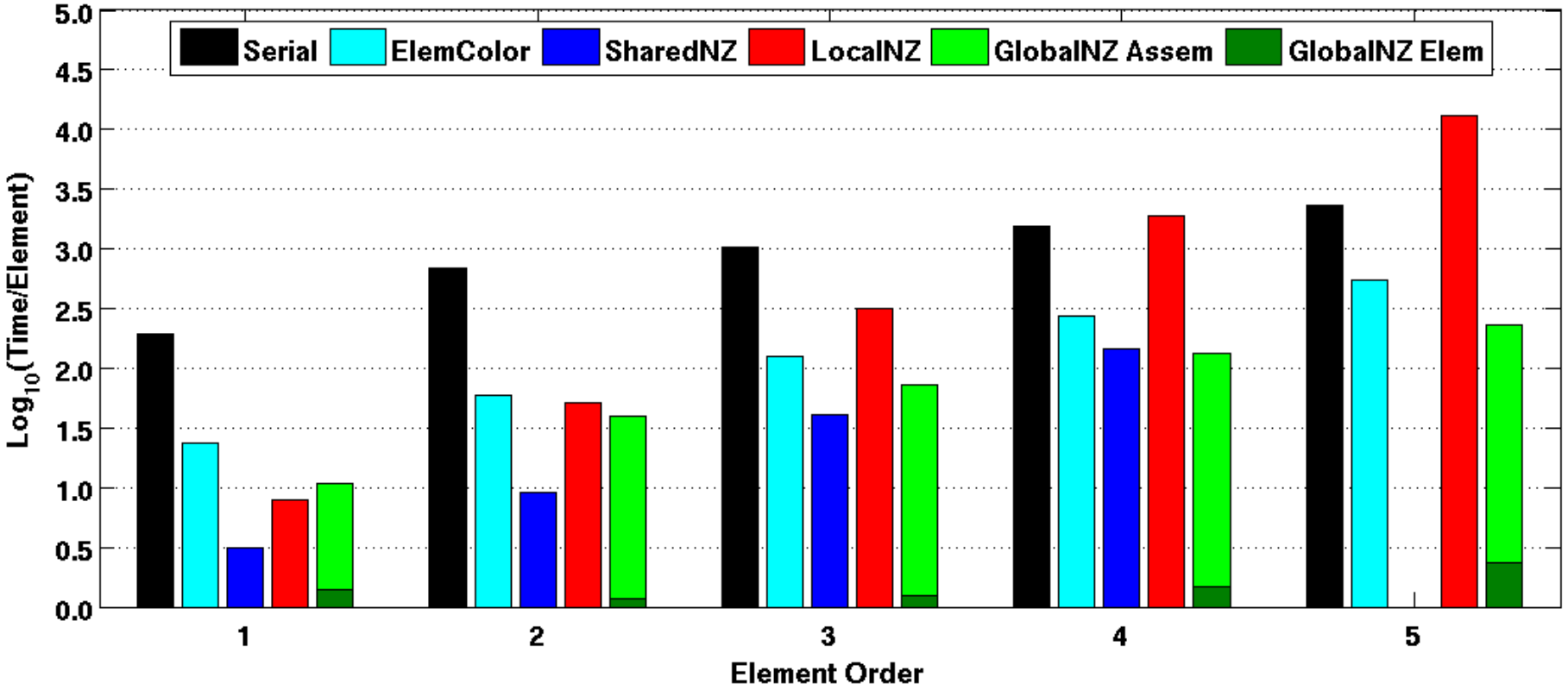


- Significant setup cost.
- Significant memory cost.

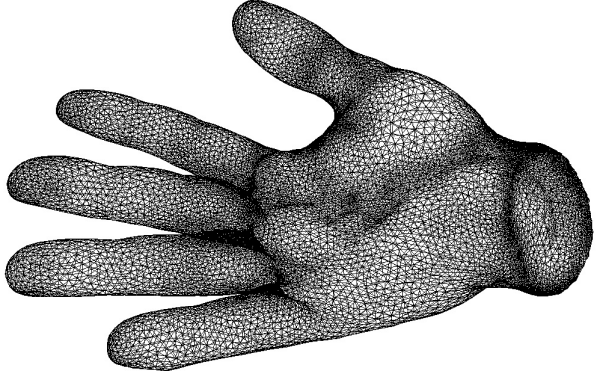
Scaling with Element Number



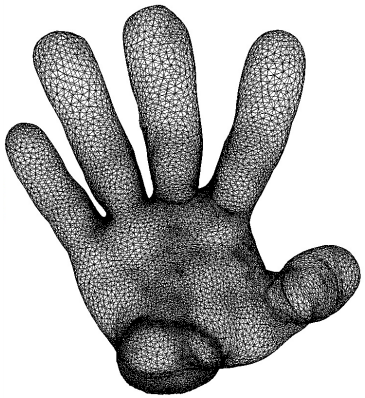
Scaling with Element Order



Application



GPU non-linear neoHookean model.



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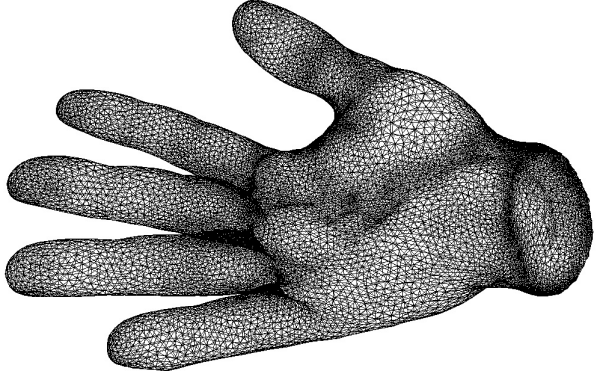

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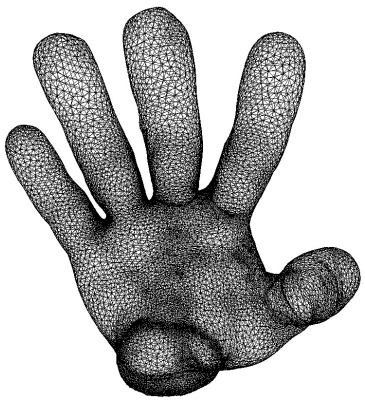

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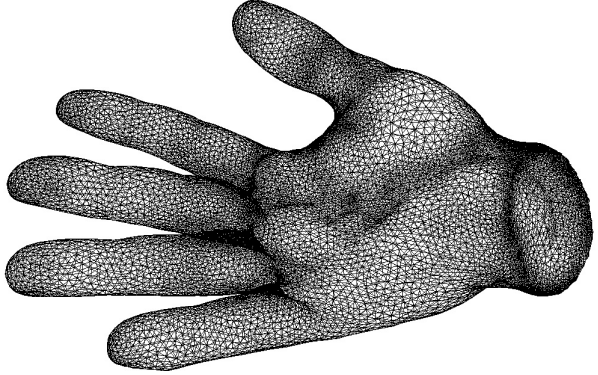

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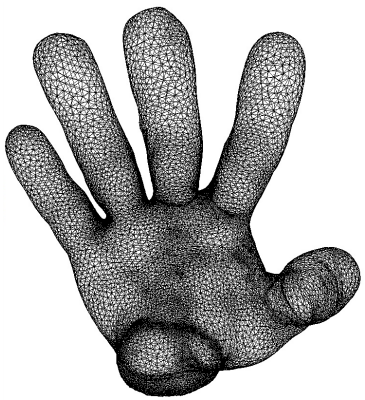

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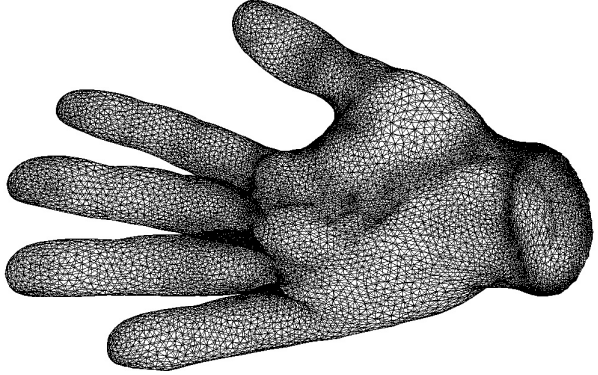

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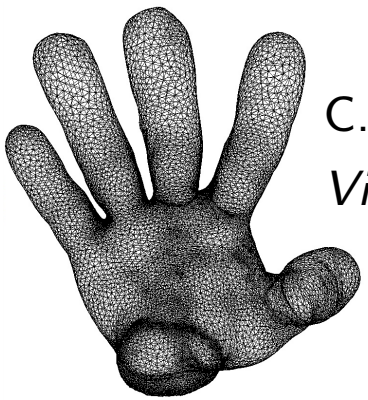
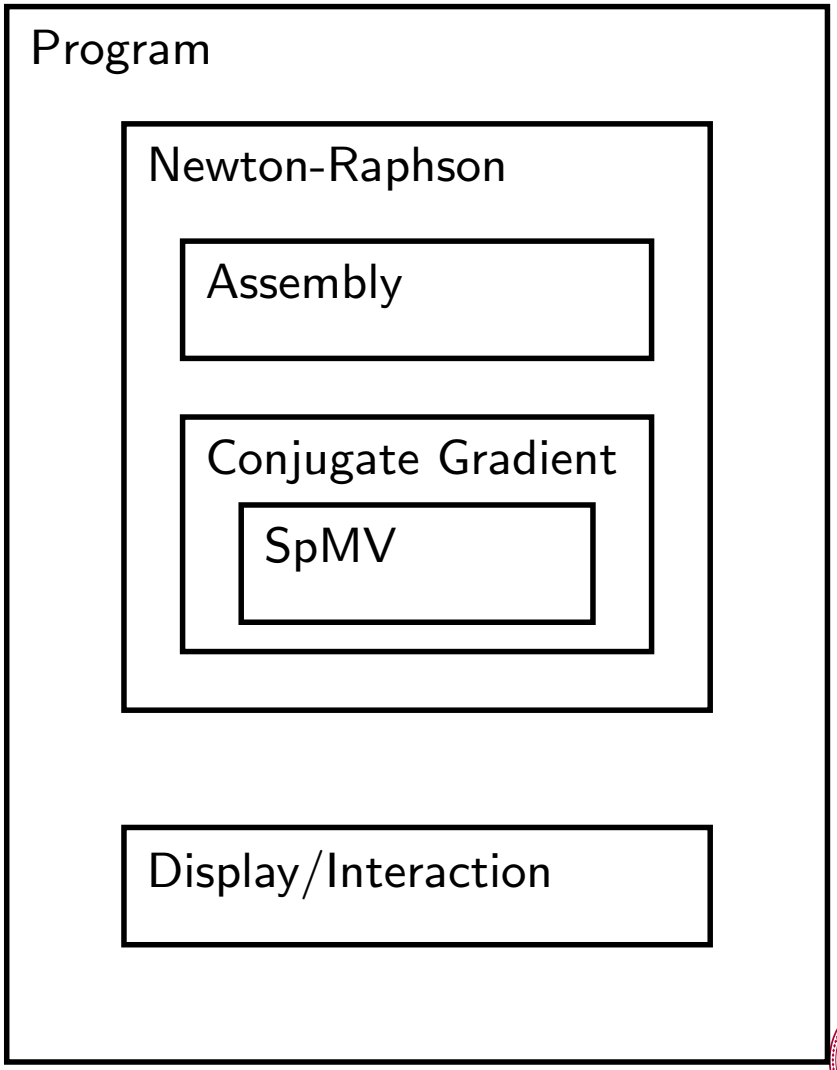


C. Cecka, A. Lew, E. Darve. *Application of Assembly, Solution, and Visualization of FEM on GPUs.* GPU Gems. Preprint.

Implementation

- Newton-Raphson
  - `NR_CPU`
  - `NR_GPU`
- Assembly
  - `AssemblyCPU`
  - `AssemblyCPU_Opt`
  - `AssemblyGlobalNZ`
  - `AssemblySharedNZ`
- Conjugate Gradient
  - `CG_CPU`
  - `CG_GPU`
  - `DCG_CPU`
  - `DCG_GPU`
- Sparse Matrix Format
  - `COO_Matrix`
  - `CSR_Matrix`
  - `HYB_Matrix`
  - `DCOO_Matrix`
  - `DCSR_Matrix`
  - `DHYB_Matrix`
- All SpMVs on CPU and GPU.



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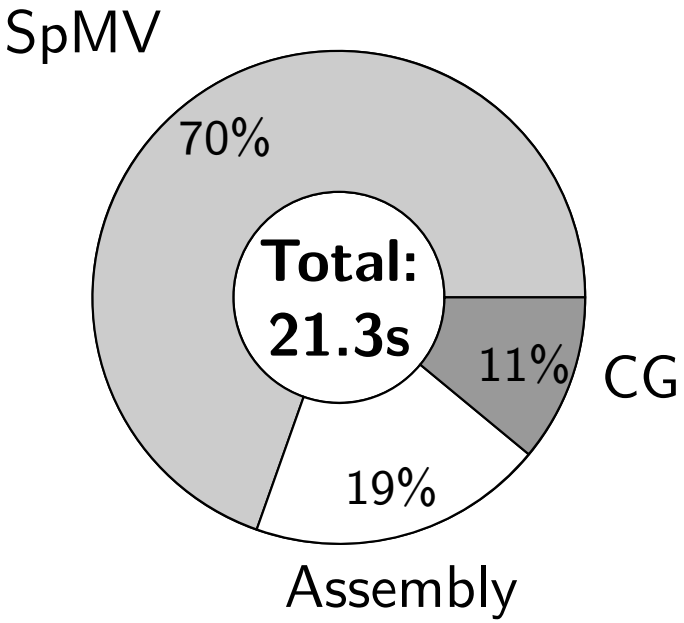

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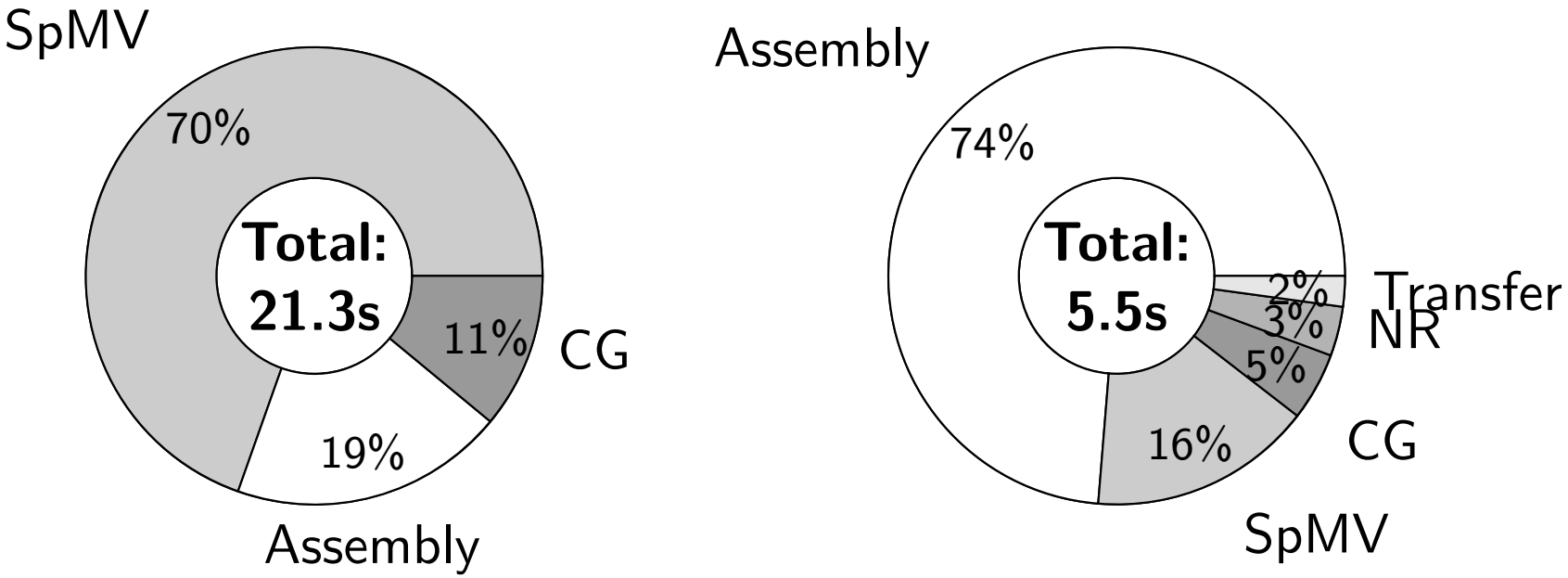

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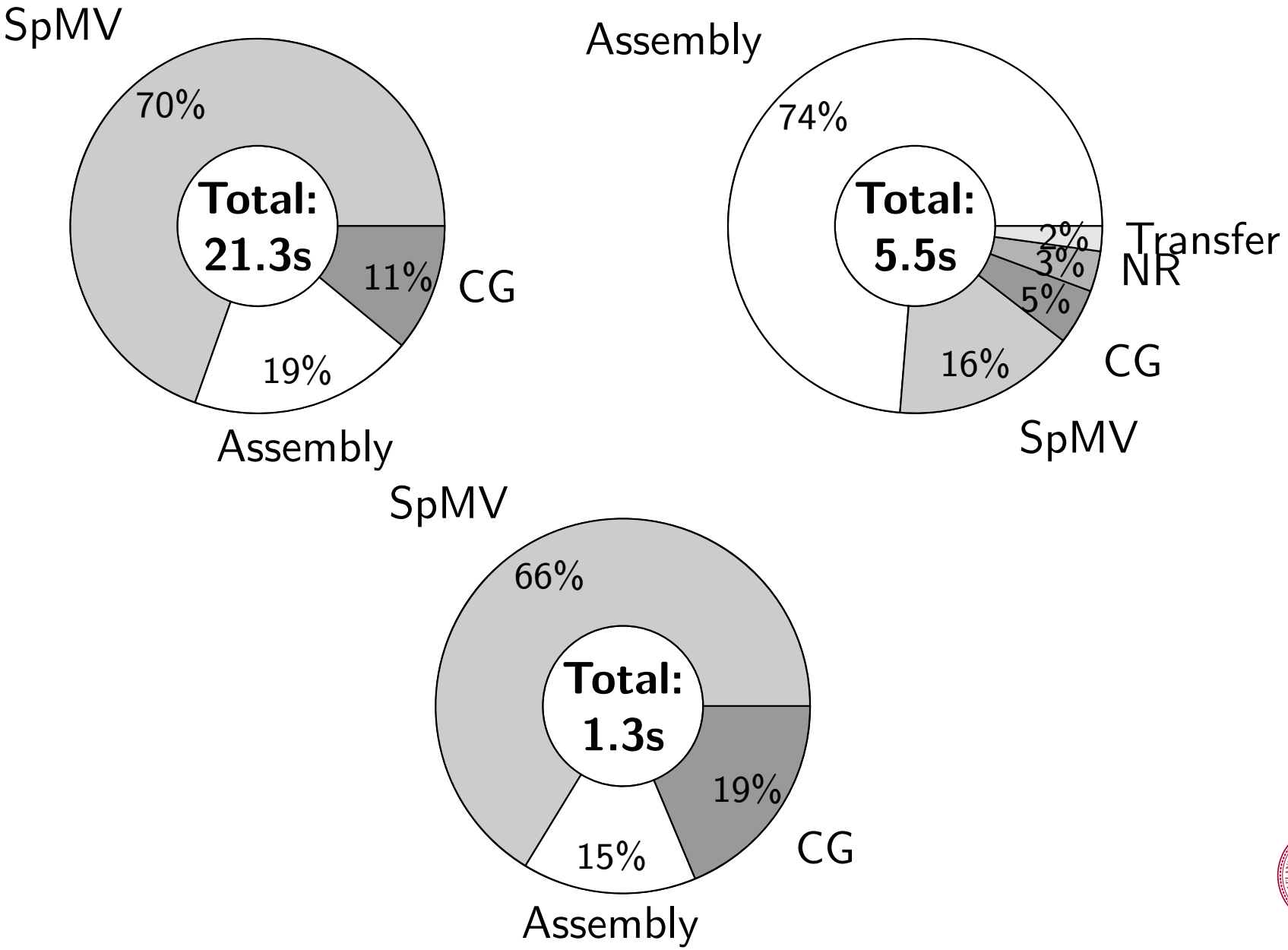
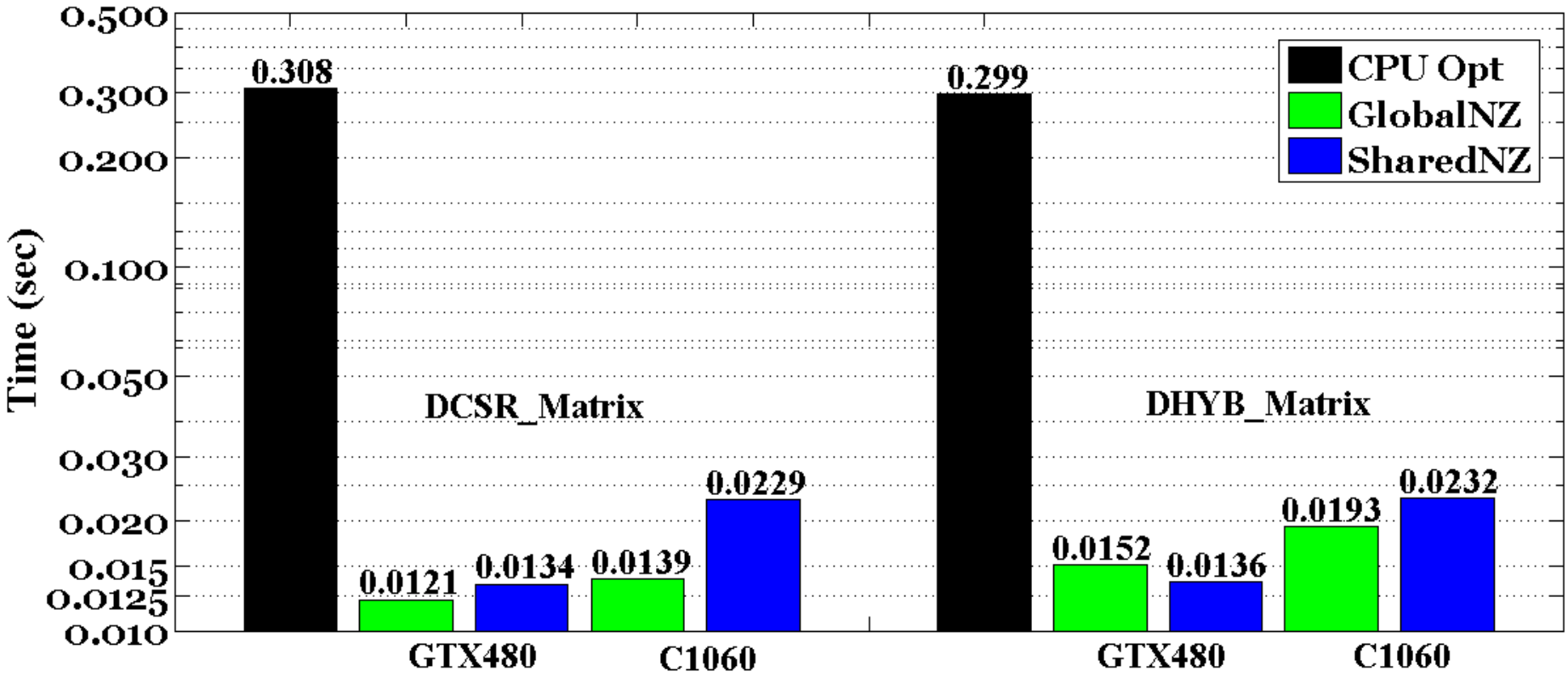


Assembly Speed



FEM Conclusion

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- Applying the methods to a high-performance FEM application.

FMM GPU Algorithms

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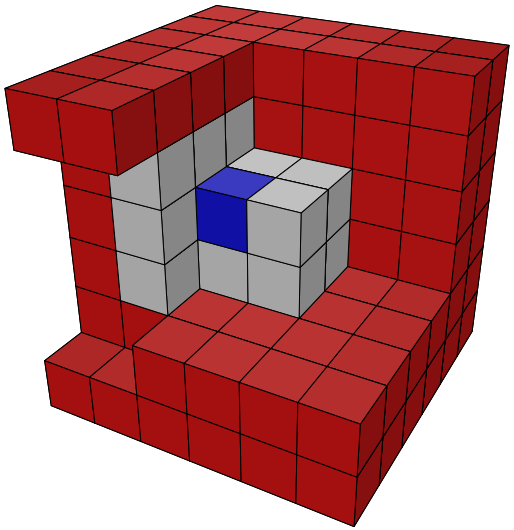
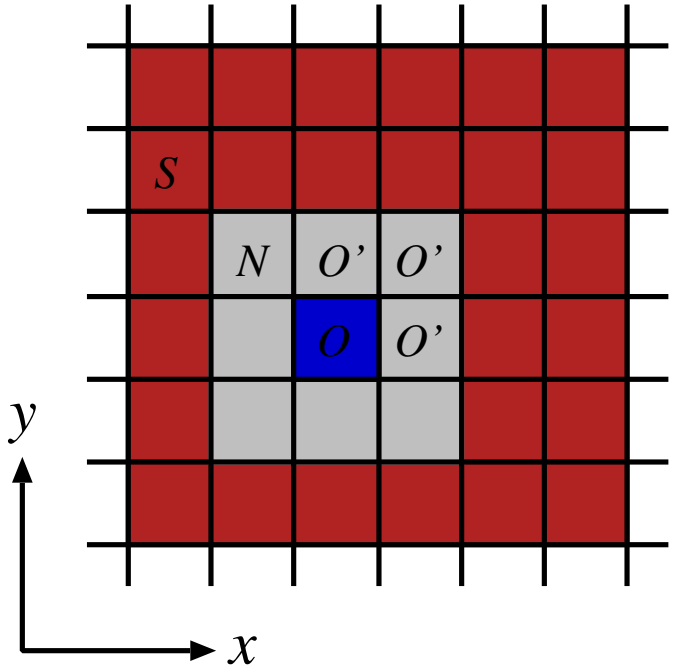
  - 316 Transfer Matrices
  - $|\mathcal{I}(O)| \leq 189$

FMM GPU Algorithms





Basic Algorithm

Block = Observation Cell O

Thread = $\mathbf{L}_i$

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    Read $\mathbf{D}^(S, O)$
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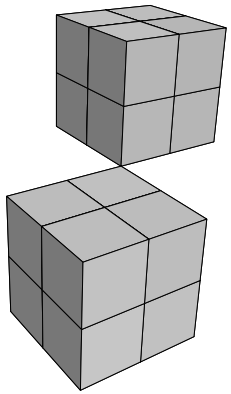
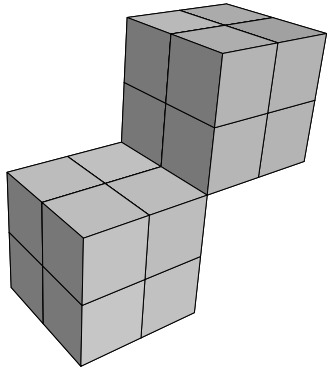
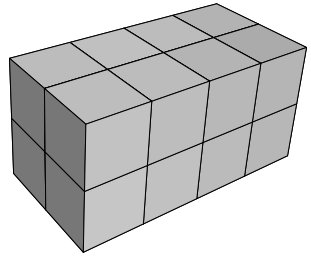
		per observation cell
Read M-data	[word]	$189r$
Read D-data	[word]	$189r^2$
Write L-data	[word]	r
Operation counts	[flop]	$189r(2r-1)$
Flop-to-word ratio	[flop/word]	1.9 for $r = 32$ $(n = 4)$
		2.0 for $r = 256$ $(n = 8)$

Blocking Siblings







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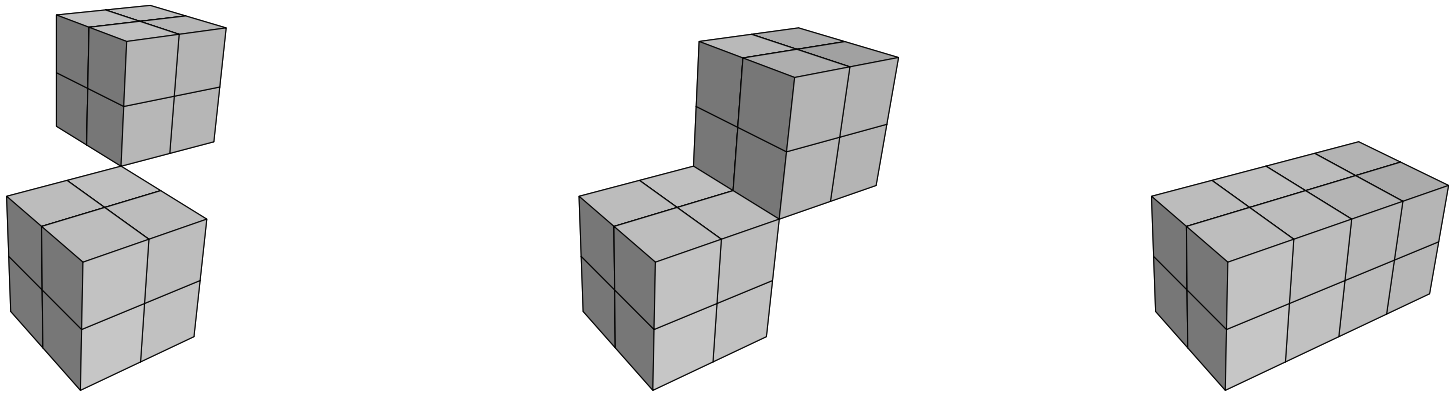
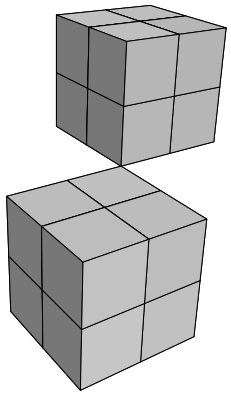
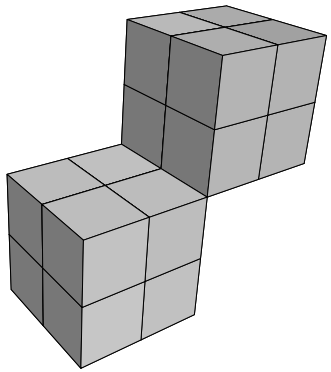
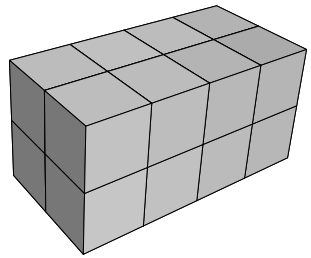


- Iterate over clusters instead of cells

		Per observation cluster	Per observation cell
Read M-data	[word]	$8 \cdot 26 \cdot r$	$26r$
Read D-data	[word]	$26 \cdot 27 \cdot r^2$	$\frac{26 \cdot 27}{8} r^2$
Read/Write L-data	[word]	$8 \cdot 26 \cdot r$	$26r$
Operation counts	[flop]	$8 \cdot 189 \cdot r(2r-1)$	$189r(2r-1)$
Flop-to-word ratio	[flop/word]	4.2 for $r = 32$ 4.3 for $r = 256$	

Blocking Siblings

- Iterate over clusters instead of cells
  - Iterate over transfer class
  - Reuse the **D** matrix

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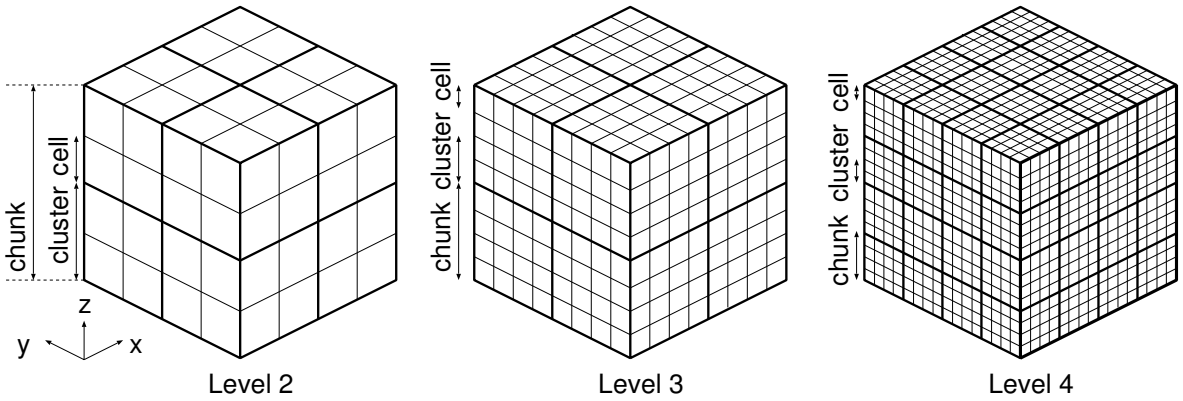

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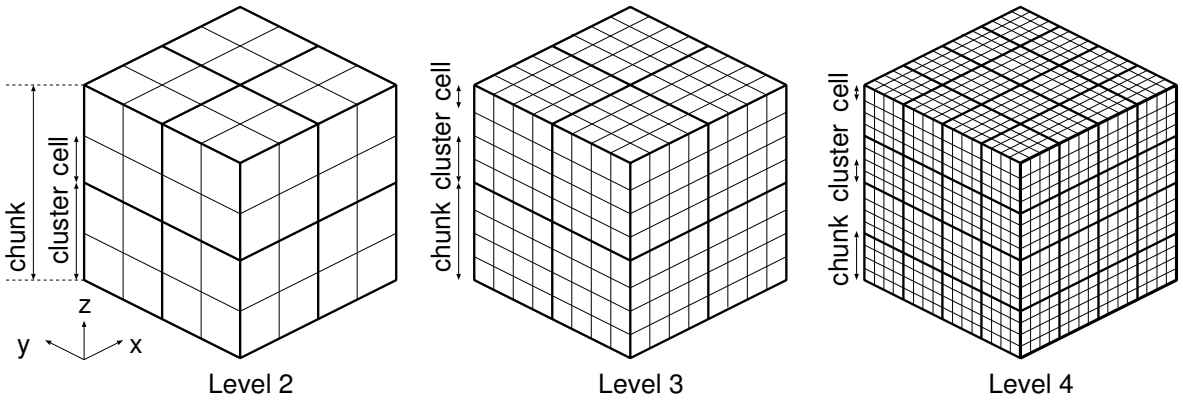

- Parallel in the observation cell
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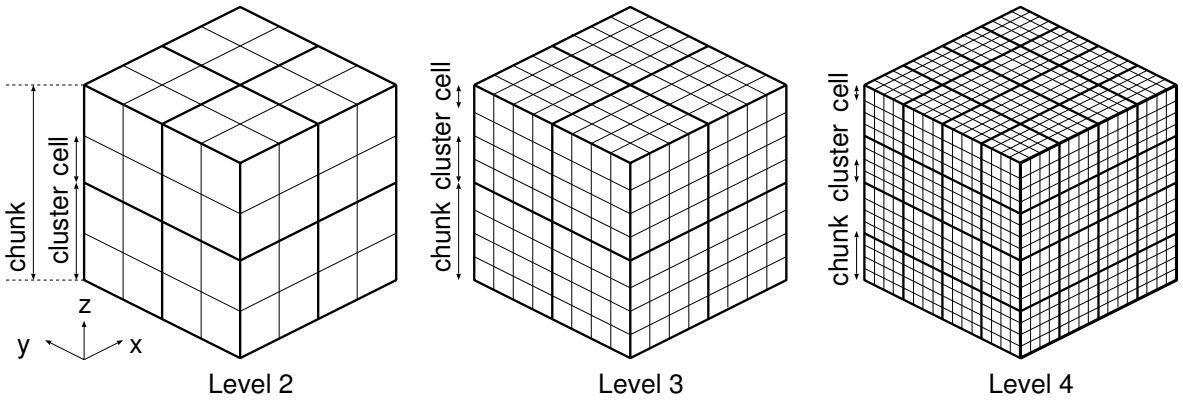

- Parallel in the observation cell
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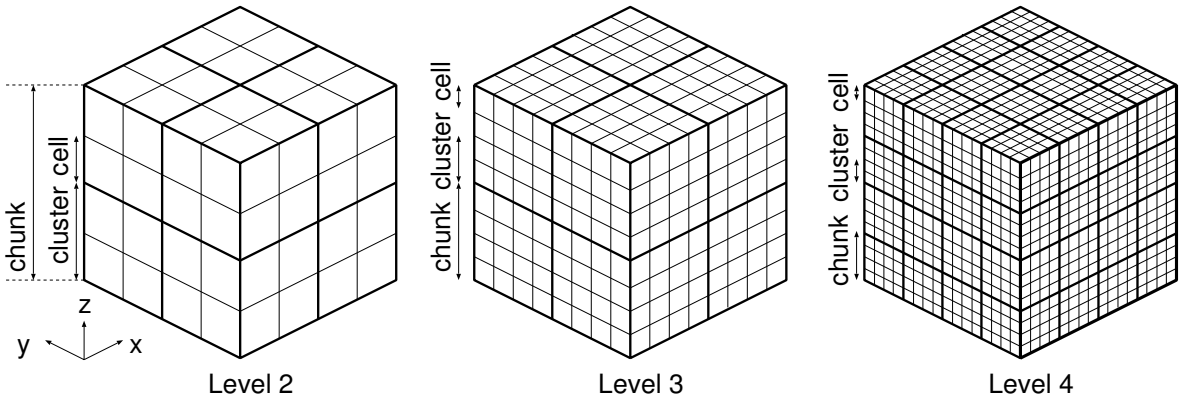

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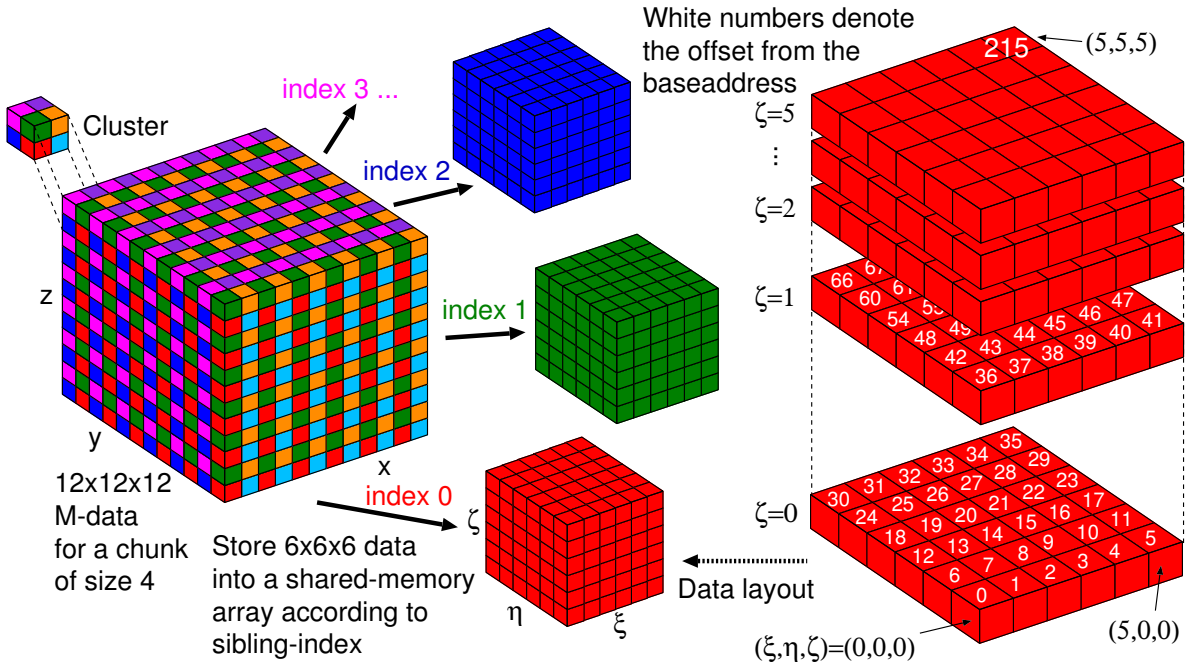

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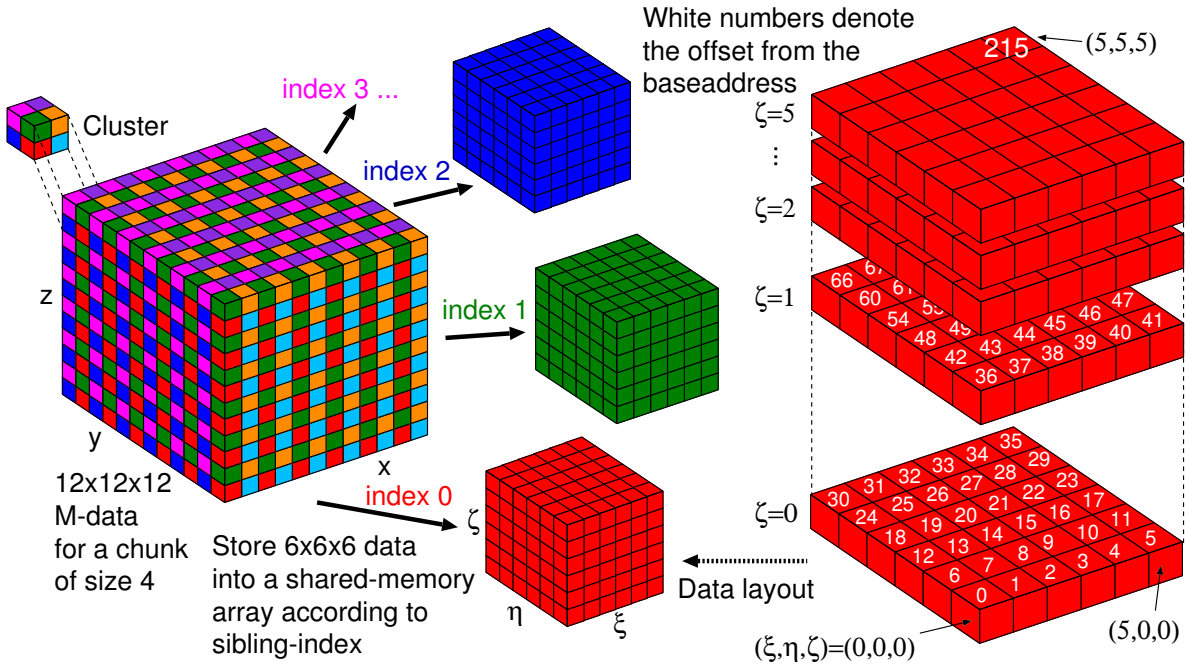

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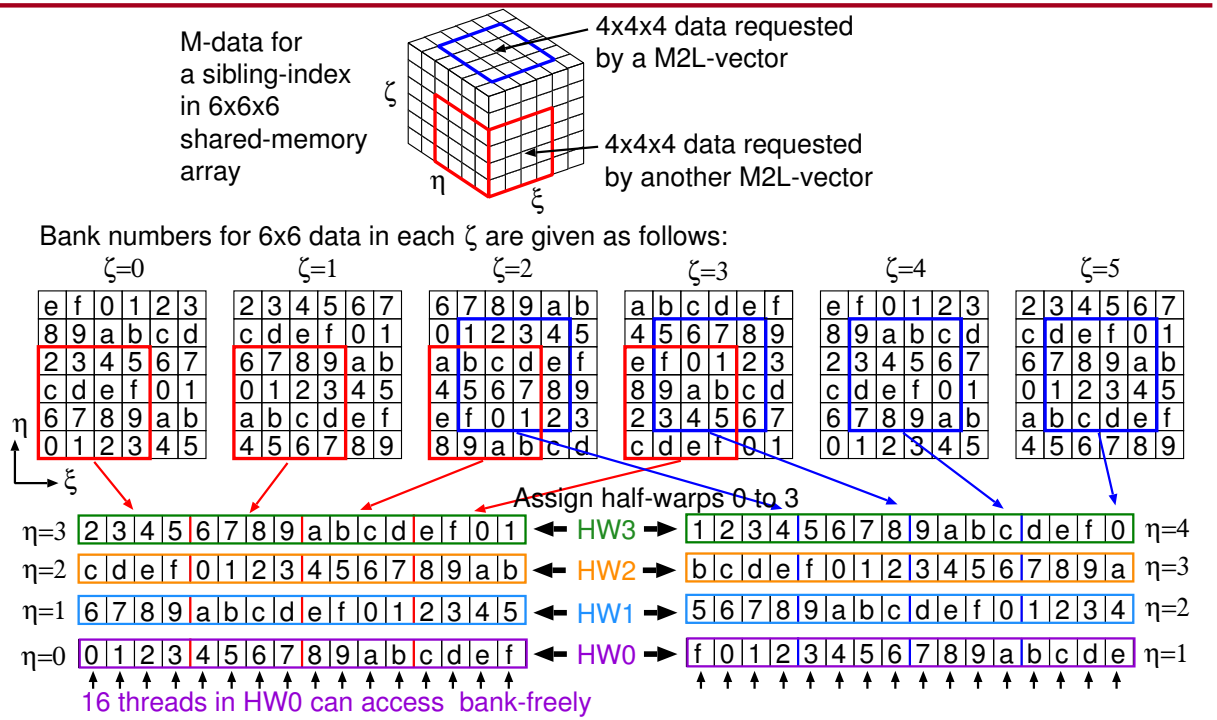

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Chunking Clusters

		per chunk of size B	per observation cell
Read M-data	[word]	$(2B+4)^3 \cdot r \cdot P$	$\frac{(B+2)^3 P}{B^3} r$
Read D-data	[word]	$316 \cdot r \cdot \frac{r}{P} \cdot P$	$\frac{316}{8B^3} r^2$
Read/Write L-data	[word]	$2 \cdot 8B^3 \cdot r \cdot \frac{r}{P} \cdot P$	$2r^2$
Operation counts	[flop]	$8B^3 \cdot 189 \cdot \frac{r}{P}(2r-1) \cdot P$	$189r(2r-1)$
Flop-to-word ratio	[flop/word]	108 for $r = 32$, $B = 4$, $P = 8$ 133 for $r = 256$, $B = 4$, $P = 16$	

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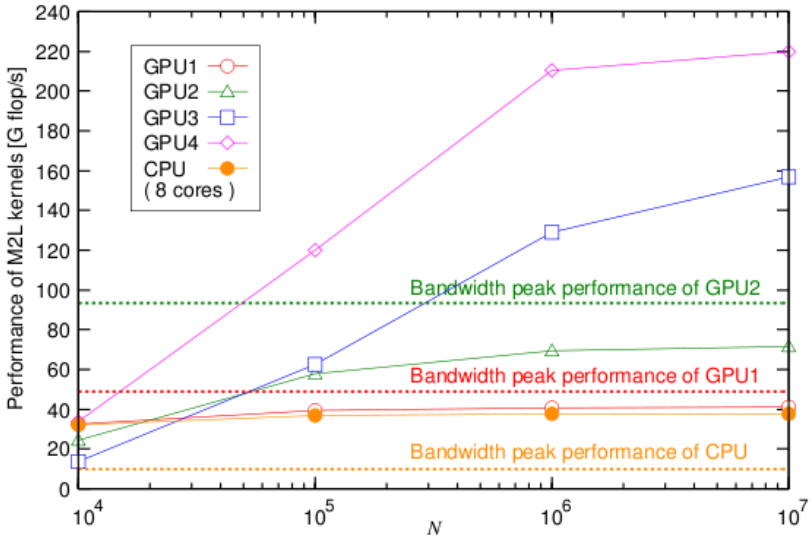
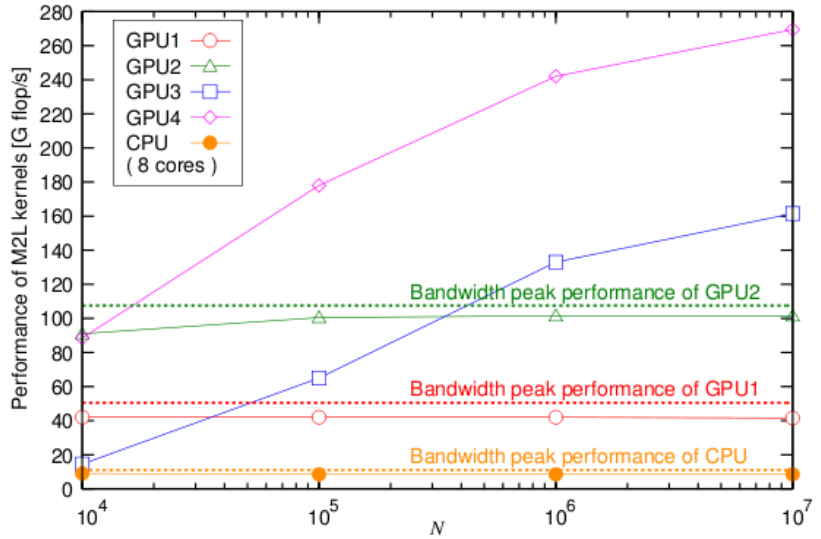
FMM Performance





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- FEM assemblies on GPU which are connectivity aware and kernel independent.
- FMM M2L computations on GPU designed to address connectivity and remain kernel independent.